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THE UBIQUITOUS DIGITAL TIME GROUP T_G

K. Srinivasa Rao[1] and Pankaj Pundir[2]

[1] Visiting Professor, IIIT, Chittoor, Sri City, Andhra Pradesh, India
& Senior Professor (Retd.), Institute of Mathematical Sciences,
C.I.T. Campus, Taramani, Chennai - 600113, India.
E-mail: ksrao18@gmail.com

[2] Research Assistant, Indian Institute of Information Technology, Chittoor,
Sri City, Andhra Pradesh - 517 588, India.
E-mail: pundir.pankaj25@gmail.com

Abstract: The digital time H : M : S is defined with three two-digit fields as $h_2h_1 : m_2m_1 : s_2s_1$, identified with appropriate restricted place values on the hour (H), minute (M) and second (S) fields, is shown to be an 86,400-element cyclic time group, T_G. A palindromic sequence of 119-elements and its sub-sequences are consequences of this group T_G.

Preamble

Time is a continuous real variable and flows smoothly. Great precision in Time measurement became important to announce Olympic world records in the realm of Sports, and essential in advanced technological operations, especially after the advent of Space Research. Historically, till the beginning of the 20th century, wrist-watches were almost exclusively worn by women, while men used pocket watches. "Officers in the British Army began using wristwatches during colonial military campaigns in the 1880s, such as during the Anglo-Burma War of 1885", David Boettcher [1]. Measurement of time using watches has been a part of a way of life for ages now. A group, $\mathbf{T_G}$, embedded in digital time reckoning, is defined here and a *palindromic sequence* derived, from the differences of a subset of elements, of the new group.

Wrist watches and wall clocks with digital dials are more common than their analogue counterparts with Arabic/Roman numerals[1]. In vogue are watches with 12-hour dials. Digital time became a necessity to specify, unambiguously, the precise time for the Arrival or Departure of the Trains and Planes, without the common A.M. and P.M. suffixes, invented to suit the 12-hour dials of wrist watches and clocks.

Let a given instant of time, specified by its hour, minute, second fields be denoted as H:M:S or more precisely as $h_2h_1 : m_2m_1 : s_2s_1$, with the restrictions on the digital values being:

$$0 \leq h_2 \leq 2,\ 0 \leq h_1 \leq 3,\ 0 \leq m_2 \leq 5,\ 0 \leq m_1 \leq 9,\ 0 \leq s_2 \leq 5,\ 0 \leq s_1 \leq 9$$

for the sequence of six digits (from L to R): $h_2, h_1, m_2, m_1, s_2, s_1$.

By definition, the first and last elements of T_G, correspond to the times 00:00:00 and 23:59:59. Let these be denoted by g_0 and g_N where $N = 86399$. Any element g_k which corresponds to a specific time, say, $h_2h_1 : m_2m_1 : s_2s_1$, is the k-th element of the group, where

$$k \ = \ 3600 \times H + 60 \times M + S \ \equiv \ 3600 \times h_1h_2 + 60 \times m_1m_2 + s_1s_2.$$

For example, 12 Noon is 12 : 00 : 00 and it is the element with index

$$k = 3600 \times 12 + 60 \times 00 + 60 \times 00 = 43200.$$

A day starts with time specified by 00:00:00 and the end of the 24-hour day is specified by the time 23:59:59. The H, M, S fields take 24, 60, 60 values, respectively, and therefore the number of elements of the group T_G is their product : $24 \times 60 \times 60 \ = \ 86400$.

$\mathbf{T_G}$ is a finite additive group, since:
(i) 00:00:00 is the identity element.
(ii) The sum of any two time elements of this group is also an element belonging to T_G. (Note: the addition of any two elements is governed by the modular nature of the H, M, S fields, which are mod 24, mod 60 and mod 60, respectively. Since $H \equiv h_2h_1, M \equiv m_2m_1, S \equiv s_2s_1$, with restrictions on the domains as pointed out

[1]On the occasion of the 125th Birth Anniversary of Srinivasa Ramanujan, the Indian Institute of Science Education Research (IISER), Pune, India, brought out a wall clock with a novel dial – viz. a dial with the following sequence, in place of the traditional Arabic/Roman numerals for one to twelve: $-\mathbf{e}^{\mathbf{i}\pi}$, $\sqrt{\pi^\pi/\pi}$, π, $\pi! - \pi$, $\sqrt{\mathbf{e}^\pi}$, $\sqrt{\pi^\pi}$, $\pi!$, $\mathbf{e}^\pi - \pi$, $\pi^{\sqrt{\pi^\pi}/\pi}$, $\pi \times \pi$, $\mathbf{e}^\pi$, $\pi^{\mathbf{e}} + (\pi \times \pi)/\sqrt{\pi^\pi}$.

earlier.)
(iii) The associativity and distributivity properties are also satisfied under the special condition for modular addition in the hours, minutes and seconds fields of H:M:S, as can be trivially verified. For example, to prove associativity, let

$$t_1 = 12:37:56,\ t_2 = 14:56:29, \text{ and } t_3 = 09:38:41$$

Then, by modular addition:

$$t_1 + t_2 = 12:37:56 + 14:56:29 = 03:34:25$$

and

$$(t_1 + t_2) + t_3 \ = \ 03:34:25 + 09:38:41 \ = \ 13:13:06$$

Similarly, $t_1 + (t_2 + t_3) = 12:37:56 + 00:35:10 = 13:13:06$.
To prove distributivity, $t_1 \times (t_2 + t_3) = (t_1 \times t_2) + (t_1 \times t_3)$,
let $t_1 = 12:34:56,\ t_2 = 01:02:03,\ t_3 = 04:05:06$, then

$$t_1 \times (t_2 + t_3) = (12:34:56) \times (05:07:09) = (16:06:24) \text{ and}$$

$$t_1 \times t_2 + t_1 \times t_3 = (12:34:56) \times (01:02:03) + (12:34:56) \times (04:05:06)$$
$$= (13:14:48) + (02 + 55:36) = (16:06:24),$$

by the definition of modular addition.
(iv) The inverse of a given element is defined as follows: Since g_0 is the first element corresponding to 00:00:00 and g_N the last element corresponding to the 23:59:59. If $t = H : M : S$, then the inverse of the element $H : M : S$ is $t^{-1} = (23 - H) : (59 - M) : (60 - S)$. For example, if $t = 12 : 34 : 56$ is the 45296-th element, by definition, and its inverse is: $t^{-1} = 11 : 25 : 04$, the 41104-th element of T_G and their sum is 24:00:00, which by modular addition is the identity 00:00:00, and the index of the inverse of the k-th element is given by:

$$(86400 - k)\ mod\ 86400, \quad \text{for } 0 \leq k \leq 86399.$$

This is true for all H:M:S. The two elements 00:00:00 and 12:00:00 are self-inverses. (The 24-hour day is represented on a 12-hour dial on ordinary analogue watches. This results in a specific time belonging to the Forenoon and Afternoon being designated as A.M. and P.M.)
The 5025-th time element $01 : 23 : 45 \equiv 1 : 23 : 45$, can be treated as the first element of a new S_5 group, S_5^{new}, and the 20601-th time element as the last element of this group, corresponding to $05 : 43 : 21 \equiv 5 : 43 : 21$. The 'new' feature of this digital time symmetric group, is the presence of the colon demarcation between the H, M, S fields.

There exists, due to the natural time ordering of the elements of this cyclic digital time symmetric group, an ordering. We define the 'distance' between adjacent time ordered elements of this group, to select a set of 119 elements from T_G. The elements of this group: S_5^{new}, a subset of T_G, generated by all permutations of the form $H : M : S$ composed of the digits 0, 1, 2, 3, 4, 5 with 0 at the leftmost position (suppressed). Unlike the normal S_5 this S_5^{new} has some distinct features:
(a) The first and last elements of this group refer to the 5,025-th and 20,601-th elements of T_G, and occur at aperiodic distances. The first element when written as (1:23:45) represents in the standard notation of the symmetric group

$$\begin{pmatrix} 1:23:45 \\ 1:23:45 \end{pmatrix} \equiv (1:23:45),$$

which is the identity element of the new symmetric group S_5^{new}, the only *new* feature being the presence of the colons.
(b) The spacing between the two consecutive elements is not the same as in the case of T_G, where it is the natural unit of time, 1 second.
(c) The spacing between the adjacent elements for the sequence of 120 elements of the group S_5^{new} is:

9, 41, 18, 41, 9, 422, 9, 91, 27, 32, 18, 413, 18, 32, 27, 91, 9, 422,
9, 41, 18, 41, 9, 1153, 9, 41, 18, 41, 9, 962, 9, 141, 36, 23, 27, 354,
18, 82, 36, 82, 18, 363, 9, 91, 27, 32, 18, 1094, 9, 91, 27, 32, 18, 363,
9, 141, 36, 23, 27, $\underline{944}$, 27, 23, 36, 141, 9,
363, 18, 32, 27, 91, 9, 1094, 18, 32, 27, 91, 9, 363, 18, 82, 36, 82,18,
354, 27, 23, 36, 141, 9, 962, 9, 41, 18, 41, 9, 1153, 9, 41, 18, 41, 9,
422, 9, 91, 27, 32, 18, 413, 18, 32, 27, 91, 9, 422, 9, 41, 18, 41, 9.

In this sequence, the central element is 944 and it is underlined.

Definition : The Length of a Palidromic sequence of elements, is the number of elements in that sequence.

The above palindromic sequence is of length 119, $\mathcal{P}_{119}$.

Discussion of the Properties of this new sequence of elements :

Definition : A sequence of elements is defined as a Palindromic sequence, when the same integer sequence occurs in the array of numbers, read from Left to Right or Right to Left.

The following points are noteworthy:

1. The spacing between the adjacent elements in the palindromic sequence $\mathcal{P}_{119}$ is of length 119 with, 18 distinct elements, as depicted above.

2. Eliminating the number 9 that occurs 24 times, a "palindormic subsequence" $\mathcal{P}_{95}$ with a reduced length of 95 elements, is obtained.

3. Continuing this reduction procedure step-by-step, we get 17 different palindromic sub-sequences from $\mathcal{P}_{119}$.

The table given below shows seventeen of the eighteen distinct elements i, used in the reduction procedure, except the singleton 944, with the corresponding frequencies and lengths of the palindromic subsequences obtained after the elimination of the recurring element, explained above.

Element i	9	41	18	422	91	27	32	413	1153	962	141
Frequency of i	24	12	18	4	8	12	8	2	2	2	4
Reduced length	95	83	65	61	53	41	33	31	29	27	23

Element i	36	23	354	82	363	1094
Frequency of i	6	4	2	4	4	2
Reduced length	17	13	11	7	3	1

Each column gives the following information about the palindromic sequence: The first column contains the numbers 9, 24, 95. These numbers stand for the fact that when the element 9, which occurs 24 times is sieved out of the palindromic sequence $\mathcal{P}_{119}$, what results is a new palindromic sub-sequence of reduced length, containing now 95 elements. In the next step when the 12 occurrences of 41 are sieved out of the 95 element sequence, we get an 83-element palindromic sub-sequence, and so on. At the end of this sieving procedure, we have the final non-trivial palindromic sub-sequence of 3 elements : 1094, 944, 1094, and the sieving reduction procedure ending in the central element 944, whose frequency is 1. Note that the i-th element which is sieved has always an even frequency. Since we started with a palindromic sequence of odd length, at each step of this sieving procedure, we obtained a palindromic subsequence with odd length. Obviously, the odd-palindromic sequence of length 119, in the reduction procedure adopted here, ends with its middle term, which is 944. However, note that, if the length of the palindromic sequence is even, and the recurring elements are sequentially sieved out, to reach, in the penultimate step, a pair of identical elements, resulting in the end element being 0, always.

The Palindromic Sequence associated with S_5^{new}

Definition. Let $S_j = \{(h_1^j : m_2^j m_1^j : s_2^j s_1^j)\}, \ \forall\ 1 \leq j \leq 120\}$, with $h_1 \neq m_1 \neq m_2 \neq s_1 \neq s_2$ represent the 120 permutations of $(1 : 23 : 45)$. In particular, $S_1 = (1 : 23 : 45)$ and $S_{120} = (5 : 43 : 21)$.

Consider two successive elements in this sequence

$$S_k = h_1^k : m_2^k m_1^k : s_2^k s_1^k \quad \text{and} \quad S_{k+1} = h_1^{k+1} : m_2^{k+1} m_1^{k+1} : s_2^{k+1} s_1^{k+1}$$

The palindromic sequence $\mathcal{P}_{119}$ is obtained by taking differences between adjacent elements of $S_{k+1} - S_k$ given by:

$$P(k) \ = \ S_{k+1} \ - \ S_k \ =$$

$$3600\times(h_1^{k+1}-h_1^k)+60\times((m_2^{k+1}-m_2^k)\times10+(m_1^{k+1}-m_1^k))+((s_2^{k+1}-s_2^k)\times10+(s_1^{k+1}-s_1^k))$$

for $1 \leq k \leq 119$. So that,

$$\mathcal{P}_{119} = \{P(k) \mid 1 \leq k \leq 119\}.$$

Observations :

1. Let

$$S_j = (h_1^j : m_2^j m_1^j : s_2^j s_1^j) \text{ and}$$
$$S_{j+1} = (h_1^{j+1} : m_2^{j+1} m_1^{j+1} : s_2^{j+1} s_1^{j+1})$$

be two consecutive elements of the sequence from the left end. Correspondingly, let

$$S_{120-j} = (h_1^{120-j} : m_2^{120-j} m_1^{120-j} : s_2^{120-j} s_1^{120-j} \text{ and}$$
$$S_{120-j+1} = (h_1^{120-j+1} : m_2^{120-j+1} m_1^{120-j+1} : s_2^{120-j+1} s_1^{120-j+1})$$

be two consecutive elements of the sequence from the right end, at the $(j+1)$-th and j-th indices, $\forall\ 1 \leq j \leq 119$. The difference between the i-th digits of each component is the same, that is,

$$(s_i)^{j+1} - (s_i)^j \ = \ (s_i)^{120-j+1} - (s_i)^{120-j},$$

$$(m_i)^{j+1} - (m_i)^j \ = \ (m_i)^{120-j+1} - (m_i)^{120-j+1},$$

for all $1 \leq i \leq 2$, and

$$(h_i)^{j+1} - (h_i)^j \ = \ (h_i)^{120-j+1} - (h_i)^{120-j}, \quad \text{for} \quad i = 1,$$

as the differences are taken lexicographically from both the ends.

Theorem: Sequence $\mathcal{P}_{119}$ is Palindromic.

Proof. The difference between two successive elements of S_j, from left end, at j and $(j+1)$-th indices, is given by,

$$3600\times(h_1^{j+1}-h_1^j)+60\times((m_2^{j+1}-m_2^j)\times 10+(m_1^{j+1}-m_1^j))+((s_2^{j+1}-s_2^j)\times 10+(s_1^{j+1}-s_1^j)),$$

for all $1 \leq j \leq 119$. Similarly, the difference between two successive elements of S_j from right end at j and $(j+1)$-th indices, is given by,

$$3600 \times (h_1^{120-j+1} - h_1^{120-(j+1)+1})$$
$$+ 60 \times ((m_2^{120-j+1} - m_2^{120-(j+1)+1}) \times 10 + (m_1^{120-j+1} - m_1^{120-(j+1)+1}))$$
$$+ ((s_2^{120-j+1} - s_2^{120-(j+1)+1}) \times 10 + (s_1^{120-j+1} - s_1^{120-(j+1)+1})),$$

for all $1 \leq j \leq 119$.

By Observation-1, these two quantities are the same. Therefore, we get the same number from both ends when we take the difference between two successive permutations of the new permutation group S_5^{new}. Consequently, the resultant sequence of length 119, is a Palindromic sequence [2].

2. Restricting the time only to $(m : s_1 s_2)$, and allowing the three numbers to take only the values 1, 2, 3, will result in a sequence

$$(1:23),\ (1:32),\ (2:13),\ (2:31),\ (3:12),\ (3:21),$$

and consequently, the differences between successive elements, in seconds, will give rise to :

$$9,\ 41,\ 18, 41,\ 9$$

a palindromic sequence of five elements, $\mathcal{P}_5$. The same palindromic sequence is also obtained for $(\ell, \ell+1, \ell+2)$, where $0 \leq \ell \leq 3$.

3. Restricting the time only to $(m_1 m_2 : s_1 s_2)$, and allowing the four numbers to take only the values 1, 2, 3, 4 will result in a sequence

$$(12:34),\ (12:43),\ (13:24),\ (13:42), \cdots,\ (42:13),\ (42:31),\ (43:12),\ (43:21)$$

and consequently, the differences between successive elements, in seconds, will give rise to the palindromic sequence :

$$9,\ 41,\ 18, 41,\ 9,\ 422,\ 9,\ 91,\ 27,\ 32,\ 18,\ 413,$$

$$18,\ 32,\ 27,\ 91,\ 9,\ 422,\ 9,\ 41,\ 18, 41,\ 9$$

a palindromic sequence of 23 elements, $\mathcal{P}_{23}$. The same palindromic sequence is also obtained for $(\ell, \ell+1, \ell+2, \ell+3)$, where $0 \leq \ell \leq 2$.

4. Sieving of an element which recurs in a given palindromic sequence obtained above, results in a palindromic sequence of reduced length – the reduction in length being equal to the frequency of occurrence of the sieved element.

Finally, the digital time palindromic sequences are unique and the periodicity of the recurring elements is such that, sieving out these recurring digits sequentially results in reduced length palindromic sequences. Explicitly, in the above palindromic sequence, sieving out 9, whose frequency of occurrence is 6, gives rise to the palindromic sequence of length 17, which is given below:

$$41,\ 18,\ 41,\ 422,\ 91,\ 27,\ 32,\ 18,\ \underline{473},\ 18,\ 32,\ 27,\ 91,\ 422,\ 41,\ 18,\ 41$$

Sieving out 41 from this sequence, results in the palindromic sequence of length 13, due to the frequency of the occurrence of 41 being four :

$$18,\ 422,\ 91,\ 27,\ 32,\ 18,\ \underline{473},\ 18,\ 32,\ 27,\ 91,\ 422,\ 18$$

Continuing this reduction procedure results in the following reduced length palindromic sequences:

$$422,\ 91,\ 27,\ 32,\ \underline{473},\ 32,\ 27,\ 91,\ 422$$

$$91,\ 27,\ 32,\ \underline{473},\ 32,\ 27,\ 91$$

$$27,\ 32,\ \underline{473},\ 32,\ 27$$

$$32,\ \underline{473},\ 32$$

and in the end of this reduction procedure, the single element 473, which happens to be the central element of the original palindromic sequence of length 23, with which the reduction procedure was started.

5. If we sieve the palindromic sub-sequences from the palindromic sequence $\mathcal{P}_{119}$, we get a new hierarchy of palindromic sub-sequences. For example if we sieve the palindromic sequence of length five : $9,\ 41,\ 18,\ 41,\ 9$, from $\mathcal{P}_{119}$, whose frequency of occurrences is 6, what results is also a palindromic subsequence of reduced length 89.

Concluding Remarks : Battery operated watches with precision quartz movements, with Liquid Crystal Display (LCD) giving place to Light Emitting Diode

(LED) digital dials, are the order of the day. The recognizable group structure in digital time embedded in it is revealed here and an interesting palindromic sequence extracted from it. The interesting consequences of the present discussions on S_5^{new} regarding the unimodality of the palindromic sequences are being extended to the case of the regular S_5 and will be reported in a future communication [2]. This work was completed during the palindromic week (6-13-16 $\rightarrow$ m:dd:yy)

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